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# Recognition of resonance type in periodically forced oscillators

H.W. Broer, S.J. Holtman\*, G. Vegter

Johann Bernoulli Institute for Mathematics and Computer Science, University of Groningen, P.O. Box 407, 9700 AK Groningen, The Netherlands

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## ABSTRACT

This paper deals with families of periodically forced oscillators undergoing a Hopf–Neïmarck–Sacker bifurcation. The interest is in the corresponding resonance sets, regions in parameter space for which subharmonics occur. It is a classical result that the local geometry of these sets in the non-degenerate case is given by an Arnol'd resonance tongue. In a mildly degenerate situation a more complicated geometry given by a singular perturbation of a Whitney umbrella is encountered. Our main contribution is providing corresponding recognition conditions, that determine to which of these cases a given family of periodically forced oscillators corresponds. The conditions are constructed from known results for families of diffeomorphisms, which in the current context are given by Poincaré maps. Our approach also provides a skeleton for the local resonant Hopf–Neïmarck–Sacker dynamics in the form of planar Poincaré–Takens vector fields. To illustrate our methods two case studies are included: A periodically forced generalized Duffing–Van der Pol oscillator and a parametrically forced generalized Volterra–Lotka system.

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## 1. Introduction

This paper studies resonant Hopf–Neïmarck–Sacker (HNS) bifurcations in families of periodically forced oscillators. In particular, we present a new procedure for detecting such bifurcations for the classical non-degenerate and a mildly degenerate case. By extending the results of our earlier work [1], which deals with families of diffeomorphisms, this procedure is reduced to checking polynomial equalities and inequalities (*recognition conditions*) in coefficients of a finite-order jet of the family at the bifurcation point. As a consequence, a practical method for handling non-trivial examples is obtained. We note that, in particular, the investigation of the mildly degenerate resonant HNS bifurcation goes beyond known results, since it corresponds to a ‘next case’ in the general program for recognizing bifurcations; cf. [2,3].

**Resonance.** We explain resonance in the general setting of families of vector fields with a periodic orbit. Such families appear in many dynamical systems, e.g., periodically forced oscillators, coupled oscillators, feed-forward coupled cell networks and high dimensional autonomous systems. We speak of resonance when oscillatory subsystems with rationally related frequencies interact with each other. This phenomenon occurs typically, though not necessarily, for parameter values near a  $p:q$ -resonant HNS bifurcation of a periodic orbit of a family of vector fields. At such a bifurcation a corresponding Poincaré map has one pair of complex-conjugate eigenvalues of the form  $e^{\pm 2\pi i p/q}$  with  $0 < |p| < q \in \mathbb{N}$

and  $\gcd(p, q) = 1$ . Under detuning of these eigenvalues by varying parameters, subharmonics corresponding to local  $q$ -periodic orbits of the Poincaré map may branch off from the periodic orbit. The regions in parameter space for which such subharmonics of order  $q$  occur are called resonance sets.

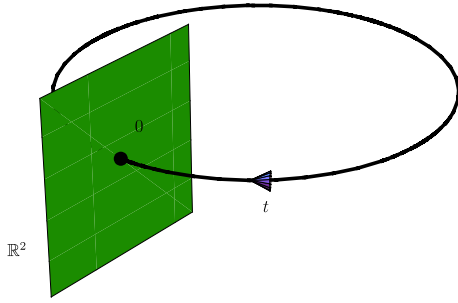
Next, we consider a vector field with a periodic orbit for which a corresponding Poincaré map has eigenvalues of the form  $e^{\pm 2\pi i p/q}$ . It is a classical result that in the case of weak resonance ( $q \geq 5$ ) a generic two-parameter deformation of such a vector field gives rise to a pair of subharmonics that appears or merges at the boundary of an Arnol'd resonance tongue in a saddle-node bifurcation [2,4,1,5]. More interestingly, for  $q \geq 7$  we encounter a mildly degenerate situation. In this case a generic four-parameter deformation yields bifurcations of up to four subharmonics near the central  $p:q$ -resonant HNS bifurcation point [4,1,6]. The local geometry of the accompanying resonance set contains cusps and swallowtails and is a singular perturbation of a Whitney umbrella; see [6].

In other words, generic families undergoing a non-degenerate  $p:q$ -resonant HNS bifurcation correspond to a class of HNS families with locally diffeomorphic resonance sets. The same holds for the mildly degenerate case. Our main result is solving the *recognition problem* belonging to this classification, i.e., we derive semi-algebraic recognition conditions for HNS families that determine the geometry of the resonance sets. These conditions also classify corresponding Poincaré–Takens reductions, which determine local dynamical features [7,8,5,9].

**Setting.** The approach explained in [12,10,3,11] reduces the study of families of vector fields in  $\mathbb{R}^k$ ,  $k \geq 3$ , with a periodic orbit for which a Poincaré map has only one pair of critical eigenvalues of

\* Corresponding author. Tel.: +31 50 363 4567; fax: +31 50 363 3800.

E-mail addresses: [h.w.broer@rug.nl](mailto:h.w.broer@rug.nl) (H.W. Broer), [s.j.holtman@rug.nl](mailto:s.j.holtman@rug.nl) (S.J. Holtman), [g.vegter@rug.nl](mailto:g.vegter@rug.nl) (G. Vegter).



**Fig. 1.** A sketch of a three-dimensional center manifold, which is a (reduced) tubular neighborhood of a periodic orbit. The coordinate  $t$  acts as an angular coordinate [10,11].

the form  $e^{\pm 2\pi i p/q}$  to the study of periodically forced oscillators. More specifically, we may restrict ourselves to a three-dimensional center manifold given by a reduced tubular neighborhood of a periodic orbit (see Fig. 1), which can be obtained by considering moving coordinate systems and appropriate time and coordinate transformations. The resulting families are of the form

$$\begin{cases} \dot{y} = h_\mu(y) + \varepsilon H_\mu(y, t), \\ \dot{t} = 1, \end{cases} \quad (1)$$

which is a standard system form for a periodically forced oscillator. Here  $h_\mu$  and  $H_\mu$  depend smoothly on both the multi-parameter  $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{R}^n$  and the coordinates  $y \in \mathbb{C}$  and  $(y, t) \in \mathbb{C} \times \mathbb{R}/\mathbb{Z}$ , respectively (for convenience we identify  $\mathbb{R}^2$  with  $\mathbb{C}$ ). Furthermore, we assume that  $\varepsilon$  is a small positive real constant,  $h_0(0) = 0$ ,  $D_y h_0(0) = 2\pi i p/q$  and that several versality conditions are satisfied; see Section 2. Under these assumptions system (1) has a  $p/q$ -resonant HNS bifurcation point near  $\mu = 0$  [13].

For the family given in (1) we explain explicitly how the recognition conditions determine the local geometry of the resonance set and the local dynamics near the resonant HNS bifurcation. To this end we introduce the vector field  $N_\mu$ , which is the  $(q-1)$ -jet of a Poincaré–Takens reduction of (1); see [7,13,5,9].

Up to a near identity diffeomorphism, the Poincaré map of (1) near  $y = 0$  is given by

$$P_\mu(y) = \Omega_{p/q} \circ N_\mu^1(y) + O(|y|^q), \quad (2)$$

where  $\Omega_{p/q}$  is the rotation over  $2\pi p/q$  around the origin and  $N_\mu^1$  is the time-1 map of  $N_\mu$ . Note that the dependence of  $N_\mu$  on  $\varepsilon$  is not explicitly indicated, since  $\varepsilon$  is a constant.

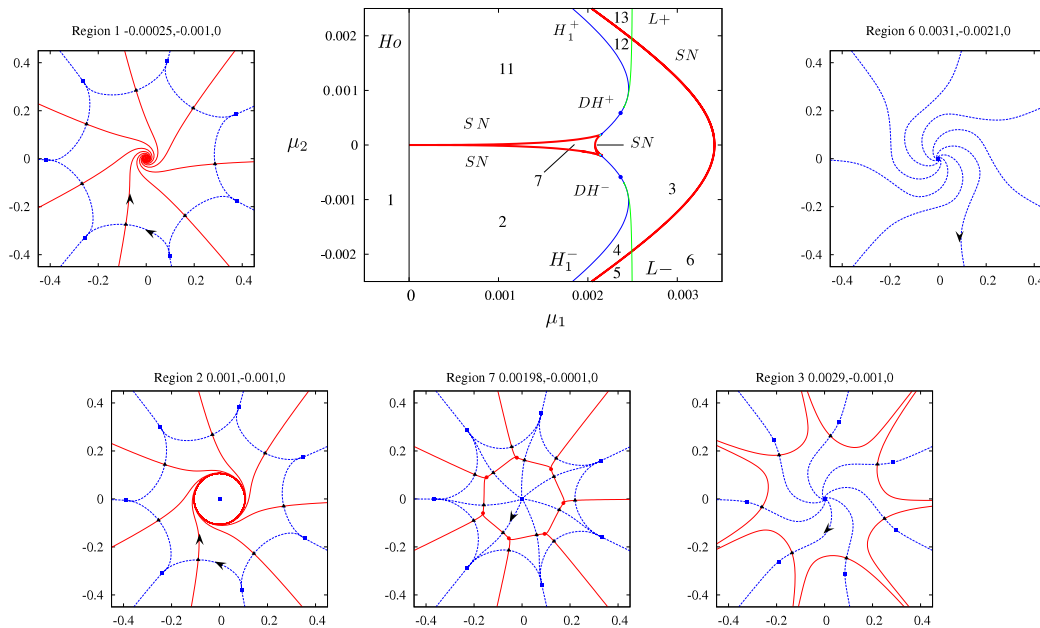
If the non-degenerate or mildly degenerate case is considered, then the local geometry of the resonance set of a HNS family of diffeomorphisms, like  $P_\mu$ , is determined by the corresponding  $(q-1)$ -jet. This follows from [1], where recognition conditions determining the geometry are provided in terms of the coefficients of this jet. Our main result, Theorem 1, gives the recognition conditions for (1) derived from the conditions given in [1] in an algorithmic way.

Perturbation arguments, based on the theories of transversality, normally hyperbolic manifolds and bifurcations [2,14,15,3,16], indicate which dynamics and bifurcations of  $P_\mu$  are induced by those of  $N_\mu$  [7,6,8,17,5]. Local features of  $N_\mu$ , e.g., saddle-node and Hopf bifurcations, translate into corresponding phenomena of  $P_\mu$ . On the other hand, more global phenomena, e.g., heteroclinic and homoclinic bifurcations, are generically expected to yield regions in parameter space characterized by tangles [2,15,17,3,5]. Fig. 2 displays a two-dimensional cross-section of the bifurcation diagram of the normal form for planar Poincaré–Takens vector fields in the mildly degenerate case. For more details see [6].

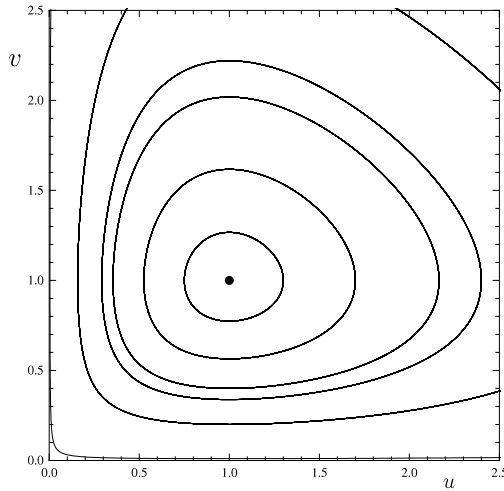
Case studies. As one case study we consider a periodically forced generalized Duffing–van der Pol oscillator [18,19] given by

$$\ddot{u} + (v_1 + v_3 u^2)\dot{u} + v_2 u + v_4 u^3 + u^5 = \varepsilon(1 + u^6) \cos(2\pi t), \quad (3)$$

where  $u \in \mathbb{R}$ ,  $\varepsilon$  is a small positive real constant and  $v = (v_1, v_2, v_3, v_4)$  is a multi-parameter in  $\mathbb{R}^4$ . Including the terms of degree 5 and 6 in  $u$  turns out to be one of the simplest generalizations of the standard Duffing–van der Pol oscillator ensuring that (3) does not exhibit too degenerate resonance phenomena (see Remarks 2 item 3). Duffing type oscillators arise in models of forced vibrations of buckled beams and electrical circuits [20]. Van der Pol



**Fig. 2.** [6] An illustrative two-dimensional cross-section of the bifurcation set of the Poincaré–Takens normal form vector field family in the mildly degenerate case surrounded by corresponding phase portraits. Here  $H_0$ ,  $SN$ ,  $L$  and  $(D)H$  denote Hopf, saddle-node, limit cycle saddle-node and (degenerate) heteroclinic bifurcations, respectively. The following typographical conventions are used in the phase portraits. Attracting (repelling) periodic orbits are represented by solid (dashed) curves. Similarly, unstable (stable) manifolds of saddle equilibria are solid (dashed) curves. Equilibria are also plotted differently according to their stability properties: attractors, repellers and saddles are plotted with small disks, squares and triangles, respectively. For more details see [6].



**Fig. 3.** Phase portrait of the Volterra–Lotka system given in (4) for  $\varepsilon = 0$  and  $(\kappa_1, \kappa_2, \kappa_3, \kappa_4) = (0, 0.8, 0, 0)$ .

type oscillators also originate from electrical circuits [21]. The coupling of the two systems has been studied a lot; see [18,19,22] and references therein, where the focus usually is on the unperturbed system ( $\varepsilon = 0$ ) or on strong resonances. For  $q = 5$  and  $q = 7$  we recover the non-degenerate HNS bifurcations, as indicated in [20,23] for periodically forced Duffing oscillators. For  $q = 7$  system (3) also exhibits the novel mildly degenerate resonant HNS bifurcation.

The second case study is a parametrically forced generalized Volterra–Lotka system [24] given by

$$\begin{cases} \dot{u} = u(-\kappa_1(1-u) + 1 - v + \kappa_3(1-v)^3 + \varepsilon \cos(2\pi t)), \\ \dot{v} = v(-\kappa_2(1-u) + \kappa_4(1-v)^3), \end{cases} \quad (4)$$

where  $(u, v) \in \mathbb{R}^2$  and  $\varepsilon$  is a small positive real constant. Moreover, here  $\kappa = (\kappa_1, \kappa_2, \kappa_3, \kappa_4)$  is a multi-parameter in  $\mathbb{R}^4$ . The form of (4) is chosen in such a way that  $(u, v) = (1, 1)$  is an equilibrium for  $\varepsilon = 0$ ; see Fig. 3. Like (3), (4) also gives rise to both a non-degenerate and a mildly degenerate HNS bifurcation. We note that parametrically forced Volterra–Lotka type systems typically model dynamics of predators ( $v$ ) and their prey ( $u$ ) in an ecosystem with, e.g., a seasonal forcing [25,24].

**Outline of the paper.** Section 2 presents the main result: The recognition conditions for non-degenerate and mildly degenerate resonant HNS bifurcations of families of the form (1). In Section 3 we apply these results to the case studies. Section 4 forms a conclusion.

**Related work.** Periodically forced oscillators are a typical example of systems of the form (1) and they have been studied a lot; see [15,3,11] and references therein. In particular V.I. Arnol'd has contributed to the general theory of resonances in this type of systems; see [26,27,2].

Subharmonics and their bifurcations are usually investigated by computing a Poincaré map. This map can be obtained by an averaging procedure involving near identity transformations [2,28,11], which is also our approach. Alternatively, continuous systems can be averaged by applying variational equations; however, by [23] this method is less suited for highly nonlinear systems. Both procedures allow one to investigate the local dynamics. For background on the study of weak and strong resonances in oscillators, see [4,7,13,17,29,5]. For resonance studied in Hamiltonian or reversible settings, see [8,30].

In [13] resonance is studied in feed-forward cell networks. Moreover, in the non-degenerate case recognition conditions are derived for a particular feed-forward coupled cell network by applying the same approach as here. In [2,3] similar procedures for

obtaining recognition conditions for families of the form (1) are presented. In contrast to the current approach, singularity theory is not applied in [3]. Moreover, in [2,3] only the non-degenerate case is discussed and recognition conditions are not explicitly computed.

Yet another procedure for detecting subharmonics is given by the Melnikov method; see [15]. By [23], this approach is less suited for studying corresponding bifurcations.

## 2. Solution of the recognition problem

Our main contribution is solving the recognition problem for non-degenerate and mildly degenerate resonant HNS bifurcations in terms of semi-algebraic conditions on coefficients of a finite-order jet of (1). The result extends the recognition conditions for families of (Poincaré) maps in [1] to families of periodically forced oscillators. Consequently, we provide a novel ‘quick’ method for detecting resonant HNS bifurcations in case studies dealing with families of the latter type. In this section we also show that the degeneracy of such bifurcations determines the geometry of the attached resonance set. Additionally, given the degeneracy, the local dynamics is (approximately) known; see [6].

**Setting.** Recall that the family (1), which we are investigating, is given by

$$\begin{cases} \dot{y} = h_\mu(y) + \varepsilon H_\mu(y, t), \\ \dot{t} = 1, \end{cases}$$

where  $(y, t) \in \mathbb{C} \times \mathbb{R}/\mathbb{Z}$ ,  $\mu \in \mathbb{R}^n$  is a multi-parameter and  $\varepsilon$  is a small positive real constant. Moreover, to ensure a  $p:q$ -resonant HNS bifurcation near  $\mu = 0$  we take  $h_0(0) = 0$  and  $D_y h_0(0) = 2\pi i p/q$  and we impose some further versality conditions specified below.

A complicated formulation of the recognition conditions is avoided by providing them in terms of a finite-order jet of a Poincaré–Takens reduction of (1). Since the relation between the original and reduced jet is also explicitly presented, this does give a solution for the recognition problem of system (1).

**Poincaré–Takens reduction of (1).** Let the power series expansion of the first component of (1) in terms of  $y$  be given by

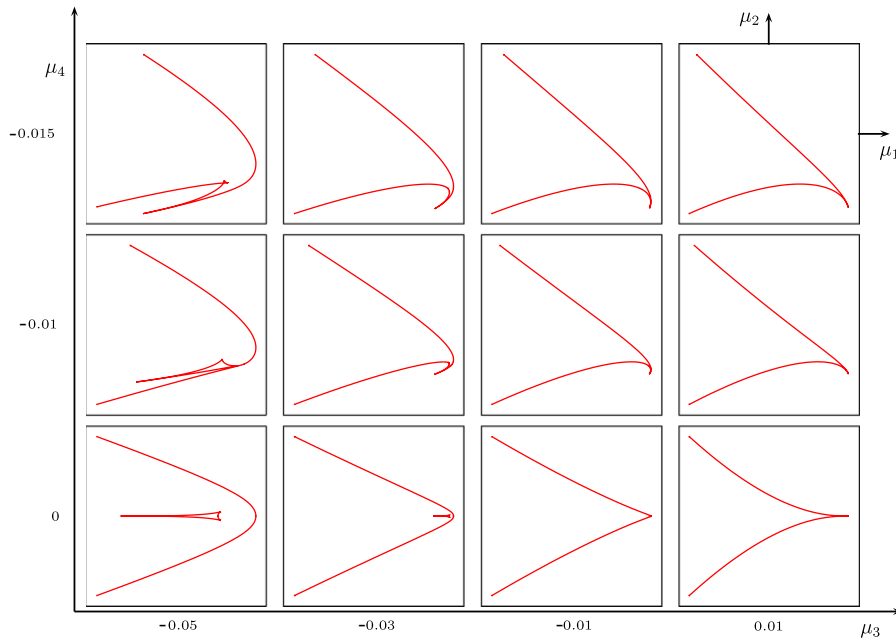
$$\begin{aligned} \dot{y} = (i\omega + h_{10}(\mu))y + \sum_{j+k=2}^{q-1} h_{jk}(\mu)y^j \bar{y}^k \\ + \varepsilon \sum_{j+k=1}^{q-1} H_{jk}(t; \mu)y^j \bar{y}^k + O(|y|^q), \end{aligned} \quad (5)$$

where  $(y, t) \in \mathbb{C} \times \mathbb{R}/\mathbb{Z}$ ,  $\mu \in \mathbb{R}^n$ ,  $h_{jk}(\mu) \in \mathbb{C}$ ,  $H_{jk}(t; \mu) \in \mathbb{C}$  and where the  $O(|y|^q)$  terms can also depend on  $t$ . Moreover, we assume that  $h_{10}(0) = 0$ ,  $\omega = 2\pi p/q$ ,  $\max\{|H_{jk}(t; 0)| \mid t \in \mathbb{R}/\mathbb{Z}, j+k \leq q-1\} > 0$  and that  $\varepsilon$  is a small positive real constant. Then a corresponding  $\mathbb{Z}_q$ -equivariant Poincaré–Takens reduced family is given by

$$\dot{z} = zG_\mu(|z|^2) + \gamma_\mu \bar{z}^{q-1} + O(|z|^q), \quad (6)$$

where  $z \in \mathbb{C}$ ,  $\gamma_\mu \in \mathbb{C}$  and where the  $O(|z|^q)$  terms may still depend on  $t$ . We explicitly derive (6) from (5) by standard techniques involving the successive application of near identity transformations removing non- $\mathbb{Z}_q$ -equivariant terms, and a Van der Pol transformation, which removes the dependence on  $t$  from the  $(q-1)$ -jet [28,4,13]. The result is that the complex-valued polynomial  $G_\mu$  is of the form

$$G_\mu(|z|^2) = \sum_{m=0}^{\lfloor \frac{q-3}{2} \rfloor} \ell_m(\mu)|z|^{2m}. \quad (7)$$



**Fig. 4.** [6] Several two-dimensional cross-sections of the discriminant set of  $\mathcal{G}_\mu$  given in (12) for  $q = 7$ , which indicate that the set contains several cusps and a swallowtail. In each of the cross-sections of  $(\mu_1, \mu_2, \mu_3, \mu_4)$ -parameter space  $(\mu_3, \mu_4)$  is fixed and  $(\mu_1, \mu_2)$  is varied.

Explicit polynomial expressions for the  $\ell_m(\mu) \in \mathbb{C}$  for  $0 \leq m \leq 2$  in terms of  $h_{jk}(\mu)$  and  $H_{kl}(t; \mu)$  with  $j + k \leq 2m + 1$  are presented up to order  $O(\varepsilon)$  in Appendix A. The dependence of  $\gamma_\mu$  on  $h_{jk}(\mu)$  and  $H_{jk}(t; \mu)$ , with  $j + k \leq q - 1$  and  $3 \leq q \leq 5$ , up to order  $O(\varepsilon^2)$  is also provided. For  $q > 5$  we refer the reader to [31]. We note that the dependence of the coefficients of the Poincaré–Takens reduction on  $\varepsilon$  is not explicitly indicated, since  $\varepsilon$  is a constant.

**Main result.** Our main result is the solution of the recognition problem for weak resonance as conditions on the coefficients of the  $(q - 1)$ -jet of (6). We observe that from Appendix A it follows that  $\ell_0(0) = h_{10}(0) = 0$  for  $\varepsilon = 0$ . Moreover, we avoid highly degenerate resonances by imposing that  $\gamma_0 \neq 0$ .

**Theorem 1** (Recognition Conditions for (5)). Consider (5) and a corresponding Poincaré–Takens reduction given in (6) with  $\ell_0(0) = 0$  and  $\gamma_0 \neq 0$ .

1. The family (5) exhibits a non-degenerate weakly resonant HNS bifurcation at  $\mu = 0$  if  $q \geq 5$ ,  $\ell_1(0) \neq 0$  and the map

$$\mu \mapsto (\operatorname{Re}(\ell_0(\mu)), \operatorname{Im}(\ell_0(\mu))) \quad (8)$$

is a submersion at  $\mu = 0$  requiring  $n \geq 2$ . This bifurcation has codimension 2. In this case the resonance set of (5) is the Cartesian product of  $\mathbb{R}^{n-2}$  and a planar  $(q - 2)/2$ -cusp (the boundary of an Arnol'd tongue). In particular, if  $n = 2$  this resonance set is locally diffeomorphic to the discriminant set of the family  $\mathcal{F}_\mu : \mathbb{C} \mapsto \mathbb{C}$  given by

$$\mathcal{F}_\mu(z) = z(\mu_1 + i\mu_2 + |z|^2) + \bar{z}^{q-1}. \quad (9)$$

Parameter values on this discriminant set satisfy

$$\mu_2 = \pm(-\mu_1)^{\frac{q-2}{2}} + O(|\mu_1|^{q-3}). \quad (10)$$

2. The family (5) exhibits a mildly degenerate weakly resonant HNS bifurcation at  $\mu = 0$  if  $q \geq 7$ ,  $\ell_1(0) = 0$ ,  $\ell_2(0) \neq 0$  and the map

$$\mu \mapsto (\operatorname{Re}(\ell_0(\mu)), \operatorname{Im}(\ell_0(\mu)), \operatorname{Re}(\ell_1(\mu)), \operatorname{Im}(\ell_1(\mu))) \quad (11)$$

is a submersion at  $\mu = 0$  requiring  $n \geq 4$ . This bifurcation has codimension 4. In this case the boundaries of the resonance set of

(5) are locally diffeomorphic to the Cartesian product of  $\mathbb{R}^{n-4}$  and the discriminant set of the family  $\mathcal{G}_\mu : \mathbb{C} \rightarrow \mathbb{C}$  given by

$$\mathcal{G}_\mu(z) = z(\mu_1 + i\mu_2 + (\mu_3 + i\mu_4)|z|^2 + |z|^4) + \bar{z}^{q-1}. \quad (12)$$

**Proof.** To prove Theorem 1 we first determine the boundary of the resonance set of (5), i.e., the parameter values for which the number of local  $q$ -periodic points of the corresponding Poincaré map changes. From [7,13,5] it follows that this Poincaré map  $P_\mu$  is given by (2),

$$P_\mu(y) = \Omega_{p/q} \circ N_\mu^1(y) + O(|y|^q),$$

where  $\Omega_{p/q}$  is the rotation over  $2\pi p/q$  around the origin. Moreover,  $N_\mu^1$  is the time-1 map of  $N_\mu$ , the vector field corresponding to the  $(q - 1)$ -jet of a Poincaré–Takens reduction of (5).

The boundary of the resonance set of  $P_\mu$  is diffeomorphic to the discriminant set of  $N_\mu$ . To explain this we consider the  $(q - 1)$ -jet of  $P_\mu$  denoted by  $j^{q-1}P_\mu(y)$ . In the non-degenerate and the mildly degenerate case the resonance sets of  $P_\mu$  and  $j^{q-1}P_\mu$  are locally diffeomorphic. This is shown in [1] by applying Lyapunov–Schmidt reduction and  $\mathbb{Z}_q$ -equivariant contact-equivalence singularity theory. Next, from (2) it follows that the  $q$ -periodic orbits of  $j^{q-1}P_\mu$  are given by the zeros of  $N_\mu$ , i.e.,

$$(j^{q-1}P_\mu)^q(y) = y \Leftrightarrow N_\mu(y) = 0.$$

Therefore, the boundary of the resonance set of  $P_\mu$  is locally diffeomorphic to the discriminant set of  $N_\mu$ , the set of parameter values for which the number of zeros of  $N_\mu$  changes.

It remains to show that the geometry of the discriminant set of  $N_\mu$  is determined by the recognition conditions stated in Theorem 1. In [4] it is asserted that all generic families of planar  $\mathbb{Z}_q$ -equivariant maps that are deformations of the same central degenerate map (the central singularity) yield diffeomorphic discriminant sets. Applying  $\mathbb{Z}_q$ -equivariant contact-equivalence singularity theory provides semi-algebraic conditions, which such families (versal unfoldings) satisfy. For the non-degenerate and mildly degenerate case these conditions are presented in [1, Theorem 2] and are repeated in Theorem 1. Moreover, from [4,1] it follows that  $\mathcal{F}_\mu$  and  $\mathcal{G}_\mu$  are universal unfoldings, i.e., versal unfoldings



with a minimum number of parameters (the codimension) satisfying the recognition conditions of [Theorem 1](#).  $\square$

**Remarks 1.** 1. The discriminant set of  $\mathcal{G}_\mu$  is studied in detail in [6]. To illustrate the geometry of this set we display several two-dimensional cross-sections in [Fig. 4](#).

2. For completeness, we consider the strong resonances for  $q = 3$  and  $q = 4$  as well, since these cases are also investigated for families of diffeomorphisms in [6].

For  $q = 3$  the family (5) exhibits a non-degenerate resonant HNS bifurcation at  $\mu = 0$  if  $\ell_0(0) = 0$ ,  $\ell_1(0) \neq 0$ ,  $\gamma_0 \neq 0$  and the map

$$\mu \mapsto (\operatorname{Re}(\ell_0(\mu)), \operatorname{Im}(\ell_0(\mu))) \quad (13)$$

is a submersion at  $\mu = 0$  requiring  $n \geq 2$ . This bifurcation is of codimension 2. If (5) satisfies the latter conditions, then the family has one subharmonic of order 3 for all parameter values near  $\mu = 0$ .

For  $q = 4$  the family (5) exhibits a non-degenerate resonant HNS bifurcation if  $\ell_0(0) = 0$ , the modulus (see [32])  $a \neq 0$ , 1 with  $a = |\ell_1(0)/\gamma_0|$ , and the map

$$\mu \mapsto (\operatorname{Re}(\ell_0(\mu)), \operatorname{Im}(\ell_0(\mu)), \operatorname{Re}(\ell_1(\mu))) \quad (14)$$

is a submersion at  $\mu = 0$  requiring  $n \geq 3$ . This bifurcation is of codimension 3. Moreover, in this case the boundary of the resonance set is diffeomorphic to the Cartesian product of  $\mathbb{R}^{n-3}$  and the discriminant set of the family  $\mathcal{H}_\mu : \mathbb{C} \rightarrow \mathbb{C}$  given by

$$\mathcal{H}_\mu(z) = z(\mu_1 + i\mu_2 + (a + \mu_3)|z|^2) + \bar{z}^3. \quad (15)$$

### 3. Case studies

The Duffing–van der Pol oscillator. We apply [Theorem 1](#) to a periodically forced generalized Duffing–van der Pol oscillator given in (3) and rewritten here in system form,

$$\begin{cases} \dot{u} = v, \\ \dot{v} = -(v_1 + v_3 u^2)v - v_2 u - v_4 u^3 - u^5 \\ \quad + \varepsilon(1 + u^6) \cos(2\pi t), \\ \dot{t} = 1, \end{cases} \quad (16)$$

where  $(u, v, t) \in \mathbb{R}^2 \times \mathbb{R}/\mathbb{Z}$  and  $\varepsilon$  is a small positive real constant. The real parameters of the system are  $v_1, v_2, v_3$  and  $v_4$ . Notice that if the  $t$ -component of the latter system is excluded, then (16) has an equilibrium at  $(u, v) = (0, 0)$  for  $\varepsilon = 0$ . Hence, the Poincaré map of (16) has a fixed point at  $(u, v) = (0, 0) + O(\varepsilon)$ ; cf. Section 1.

We focus on 1:7-resonance in the Duffing–van der Pol system, as this is the lowest  $q$  for which both the non-degenerate and a mildly degenerate resonant HNS bifurcation occur. The result is given in the following proposition.

**Proposition 2** (Solution of the Recognition Problem for the Duffing–van der Pol System for  $q = 7$ ). Consider the system given in (16) and the fixed point  $(u, v) = (0, 0) + O(\varepsilon)$  of the corresponding Poincaré map.

1. System (16) undergoes a non-degenerate 1:7-resonant HNS bifurcation for the following set of codimension 2:

$$\left\{ v \in \mathbb{R}^4 \mid v_1 = 0, v_2 = \frac{4}{49}\pi^2, |v_3| + |v_4| \neq 0 \right\}. \quad (17)$$

In this case, the attached boundary of the resonance set is locally diffeomorphic to the Cartesian product of a planar 5/2-cusp and  $\mathbb{R}^2$ .

2. System (16) undergoes a mildly degenerate 1:7-resonant HNS bifurcation for the following set of codimension 4:

$$\left\{ v \in \mathbb{R}^4 \mid v_1 = 0, v_2 = \frac{4}{49}\pi^2, v_3 = 0, v_4 = 0 \right\}. \quad (18)$$

In this case, the attached boundary of the resonance set is locally diffeomorphic to the discriminant set of

$$\mathcal{G}_\mu = z(\mu_1 + i\mu_2 + (\mu_3 + i\mu_4)|z|^2 + |z|^4) + \bar{z}^6. \quad (19)$$

**Remarks 2.** 1. Throughout, recognition conditions for case studies, i.e. (17), (18), (21), (22), (23), (33) and (34), are computed up to order  $O(\varepsilon)$ ; see the corresponding proofs and [Appendix A](#).

2. From [6] it follows that the bifurcation diagram of the forced Duffing–van der Pol system is (approximately) known near the parameter values corresponding to resonant HNS bifurcations given in [Proposition 2](#).

3. If the terms of degree 5 and 6 in  $u$  are removed from (16), then the parameter value  $(v_1, v_2, v_3, v_4) = (0, \frac{4}{49}\pi^2, 0, 0)$  would correspond to a more degenerate resonant HNS bifurcation than the mildly degenerate bifurcation discussed in the current paper. Indeed, instead of the Duffing–van der Pol system given in (16) we could consider

$$\begin{cases} \dot{u} = v, \\ \dot{v} = -(v_1 + v_3 u^2)v - v_2 u - v_4 u^3 - \alpha u^5 \\ \quad + \varepsilon(1 + \beta u^6) \cos(2\pi t), \\ \dot{t} = 1, \end{cases} \quad (20)$$

i.e., we can place parameters in front of the  $u^5$  and  $u^6$  terms. Then recognition conditions (17) and (18) become

$$\left\{ v \in \mathbb{R}^4 \mid v_1 = 0, v_2 = \frac{4}{49}\pi^2, |v_3| + |v_4| \neq 0, \beta \neq 0 \right\}, \quad (21)$$

and

$$\left\{ v \in \mathbb{R}^4 \mid v_1 = 0, v_2 = \frac{4}{49}\pi^2, v_3 = 0, v_4 = 0, \alpha \neq 0, \beta \neq 0 \right\}, \quad (22)$$

respectively.

4. We also consider a 1:5-resonance to demonstrate the dependence on  $q$  of the recognition conditions. In this case the periodic orbit of system (20) going through  $(u, v, t) = (0, 0, 0) + O(\varepsilon)$  undergoes a non-degenerate 1:5-resonant HNS bifurcation for the following set of codimension 2:

$$\left\{ v \in \mathbb{R}^4 \mid v_1 = 0, v_2 = \frac{4}{25}\pi^2, |v_3| + |v_4| \neq 0 \right\}, \quad (23)$$

which does not depend on  $\alpha$  or  $\beta$ . The boundary of the resonance set attached to the latter set is diffeomorphic to the Cartesian product of a 3/2-cusp and  $\mathbb{R}^2$ .

**Proof.** A proof of [Proposition 2](#) and the latter remarks consists of three steps:

1. Resonant Hopf points of the unperturbed part ( $\varepsilon = 0$ ) of the family given in (16) without the  $t$ -component need to be determined, since these correspond to resonant HNS bifurcations of the full family given in (16); see [13].
2. For parameter values near resonant Hopf points the family has to be transformed to the form (5) in order to apply [Theorem 1](#).
3. [Theorem 1](#) is applied to determine the degeneracy of the bifurcation points obtained in the first step.

Beginning with the first step, we determine a Hopf stratum of the unperturbed part ( $\varepsilon = 0$ ) of the Duffing–van der Pol system given in (16) without the  $t$ -component. To be precise, we focus on the following system:

$$\begin{cases} \dot{u} = v, \\ \dot{v} = -(v_1 + v_3 u^2)v - v_2 u - v_4 u^3 - u^5. \end{cases} \quad (24)$$

For parameter values on a Hopf stratum the trace of the linear part at an equilibrium of the latter system vanishes, while the corresponding determinant is positive [13]. Considering the equilibrium  $(u, v) = (0, 0)$  leads to the three-dimensional Hopf stratum given by

$$\{v \in \mathbb{R}^4 \mid v_1 = 0, v_2 > 0, v_3 \in \mathbb{R}, v_4 \in \mathbb{R}\}. \quad (25)$$

For these parameter values the eigenvalues of system (24) at  $(u, v) = (0, 0)$  are given by

$$\lambda_{\pm} = \pm i\sqrt{v_2}. \quad (26)$$

It follows that  $\lambda_{\pm}$  can take any value on the imaginary axis.

We focus on a HNS bifurcation of the system (16) near the following two two-dimensional sets:

$$\left\{v \in \mathbb{R}^4 \mid v_1 = 0, v_2 = \frac{4}{25}\pi^2, v_3 \in \mathbb{R}, v_4 \in \mathbb{R}\right\}$$

and

$$\left\{v \in \mathbb{R}^4 \mid v_1 = 0, v_2 = \frac{4}{49}\pi^2, v_3 \in \mathbb{R}, v_4 \in \mathbb{R}\right\},$$

which by (1) and (26) correspond to a 1:5- and a 1:7-weak resonance, respectively.

We continue with the second step, i.e., system (16) needs to be transformed to the form (5). A detailed explanation of the required transformations starting from a general periodically forced oscillator is given in Appendix B. As these transformations give rise to lengthy straightforward computations for the Duffing–van der Pol system, the result is only displayed for  $q = 5$ ; see Appendix B.

We end with the third step. Once system (16) is brought into the form (5), the coefficients of the  $(q - 1)$ -jet of the corresponding Poincaré–Takens reduction are easily deduced from the expressions in Appendix A. The result is that the coefficients  $\ell_m$  for  $m = 0, 1, 2$  of the reduced family corresponding to (16) are given by

$$\ell_0(v) = -\frac{1}{2} \left( v_1 + \frac{i}{\sqrt{(v_2)_0}} \tilde{v}_2 \right) + O(|v_1 + \tilde{v}_2|^2, \varepsilon), \quad (27)$$

$$\ell_1(v) = \frac{1}{2(v_2)_0 - 2} \left( v_3 - \frac{3i}{\sqrt{(v_2)_0}} v_4 \right) + O(v_1, \tilde{v}_2, \varepsilon), \quad (28)$$

$$\ell_2(v) = \frac{5i}{((v_2)_0 - 1)^2 \sqrt{(v_2)_0}} + O(v_1, \tilde{v}_2, v_3, v_4, \varepsilon), \quad (29)$$

where  $\tilde{v}_2 = v_2 - (v_2)_0$  and  $(v_2)_0 = \frac{4}{25}\pi^2$  or  $(v_2)_0 = \frac{4}{49}\pi^2$  depending on whether a 1:5- or 1:7-resonance is considered, respectively. For  $q = 5$  the coefficient  $\gamma_v$  is given by

$$\gamma_v = \varepsilon c_0 (c_1 v_3^2 + c_2 v_3 v_4 + c_3 v_4^2 + O(v_1, \tilde{v}_2, |v_3 + v_4|^3, \varepsilon)), \quad (30)$$

with

$$c_0 = \frac{1}{64 (\sqrt{(v_2)_0} - 1) (\sqrt{(v_2)_0} + 1)^5 (v_2)_0 ((v_2)_0 - 4\pi^2)},$$

$$c_1 = i \left( -(v_2)_0^2 + (10 + 12\pi)(v_2)_0^{3/2} + (1 - 2\pi)(v_2)_0 + (6 - 20\pi)\sqrt{(v_2)_0} + 2\pi \right),$$

$$c_2 = \left( -5(v_2)_0^2 + (47 - 34\pi)(v_2)_0^{3/2} + (5 - 10\pi)(v_2)_0 + (17 - 94\pi)\sqrt{(v_2)_0} + 10\pi \right),$$

$$c_3 = 3i((-11 + 10\pi)(v_2)_0 + 22\pi - 5),$$

where  $(v_2)_0 = \frac{4}{25}\pi^2$ . For  $q = 7$  the coefficient  $\gamma_v$  is given by

$$\gamma_v = -\varepsilon \left( \frac{i(\sqrt{(v_2)_0} + 1)}{4(\sqrt{(v_2)_0} - 1)^6 \sqrt{(v_2)_0}} + O(v_1, \tilde{v}_2, v_3, v_4, \varepsilon) \right), \quad (31)$$

where  $(v_2)_0 = \frac{4}{49}\pi^2$ . For small  $\varepsilon$  the latter coefficients of the Poincaré–Takens reduction of (16) and Theorem 1 result in Proposition 2.

We note that all coefficients of the Poincaré–Takens reduction of (20) are the same as those corresponding to (16) except  $\ell_2$ , and  $\gamma_v$  for  $q = 7$ , which are given by

$$\ell_2(v) = \frac{5i\alpha}{((v_2)_0 - 1)^2 \sqrt{(v_2)_0}} + O(v_1, \tilde{v}_2, v_3, v_4, \varepsilon),$$

and

$$\gamma_v = -\varepsilon \left( \frac{i\beta(\sqrt{(v_2)_0} + 1)}{4(\sqrt{(v_2)_0} - 1)^6 \sqrt{(v_2)_0}} + O(v_1, \tilde{v}_2, v_3, v_4, \varepsilon) \right),$$

respectively. Combined with Theorem 1, and the coefficients  $\ell_0$  and  $\ell_1$  given in (27) and (28), respectively, these expressions result in the recognition conditions (21) and (22).  $\square$

**Volterra–Lotka system.** Next, we consider the generalized Volterra–Lotka system given in (4) and rewritten here in system form,

$$\begin{cases} \dot{u} = u(-\kappa_1(1 - u) + 1 - v + \kappa_3(1 - v)^3 + \varepsilon \cos(2\pi t)), \\ \dot{v} = v(-\kappa_2(1 - u) + \kappa_4(1 - v)^3), \\ \dot{t} = 1, \end{cases} \quad (32)$$

where  $(u, v, t) \in \mathbb{R}^2 \times \mathbb{R}/\mathbb{Z}$  and  $\varepsilon$  is a small positive real constant. The real parameters of the system are given by  $\kappa_1, \kappa_2, \kappa_3$  and  $\kappa_4$ . Applying the results of Section 2 yields the following proposition for 1:7-resonant HNS bifurcations.

**Proposition 3** (Solution of the Recognition Problem for the Volterra–Lotka System for  $q = 7$ ). Consider the system given in (32) and the fixed point  $(u, v) = (1, 1) + O(\varepsilon)$  of the corresponding Poincaré map.

1. System (32) undergoes a non-degenerate 1:7-resonant HNS bifurcation for the following set of codimension 2:

$$\left\{ \kappa \in \mathbb{R}^4 \mid \kappa_1 = 0, \kappa_2 = \frac{4}{49}\pi^2, \right. \\ \left. |\kappa_3| + \left| \kappa_4 - \frac{(2\pi - 7)^3 (4\pi^2 + 49)}{106\pi (2\pi + 7)^3} \right| \neq 0 \right\}. \quad (33)$$

In this case, the attached boundary of the resonance set is diffeomorphic to the Cartesian product of a planar 5/2-cusp and  $\mathbb{R}^2$ .

2. System (32) undergoes a mildly degenerate 1:7-resonant HNS bifurcation for the following set of codimension 4:

$$\left\{ \kappa \in \mathbb{R}^4 \mid \kappa_1 = 0, \kappa_2 = \frac{4}{49}\pi^2, \kappa_3 = 0, \right. \\ \left. \kappa_4 = \frac{(2\pi - 7)^3 (4\pi^2 + 49)}{106\pi (2\pi + 7)^3} \right\}. \quad (34)$$

In this case, the attached boundary of the resonance set is diffeomorphic to the discriminant set of  $g_\mu$  given in (19).

**Proof.** We prove Proposition 3, applying the same steps as were given at the beginning of the proof of Proposition 2.

So starting with the first step, we determine a Hopf stratum of the unperturbed part ( $\varepsilon = 0$ ) of the Volterra–Lotka system given in (32) without  $t$ -component. More precisely, we focus on

$$\begin{cases} \dot{u} = u(-\kappa_1(1-u) + 1 - v + \kappa_3(1-v)^3), \\ \dot{v} = v(-\kappa_2(1-u) + \kappa_4(1-v)^3). \end{cases} \quad (35)$$

Considering the equilibrium  $(u, v) = (1, 1)$  of the latter system leads to the three-dimensional Hopf stratum given by

$$\{\kappa \in \mathbb{R}^4 \mid \kappa_1 = 0, \kappa_2 > 0, \kappa_3 \in \mathbb{R}, \kappa_4 \in \mathbb{R}\}. \quad (36)$$

For these parameter values the eigenvalues of system (35) at  $(u, v) = (1, 1)$  are given by

$$\lambda_{\pm} = \pm i\sqrt{\kappa_2}. \quad (37)$$

We focus on a HNS bifurcation of the system (32) near the following two-dimensional set:

$$\left\{ \kappa \in \mathbb{R}^4 \mid \kappa_1 = 0, \kappa_2 = \frac{4}{49}\pi^2, \kappa_3 \in \mathbb{R}, \kappa_4 \in \mathbb{R} \right\},$$

which by (37) and (1) corresponds to a 1:7-weak resonance.

We continue with the second step, i.e., system (32) needs to be transformed to the form (5). The required transformations are explained in Appendix B. We omit the details of the computations here, since they are long but straightforward.

We end with the third step, the  $(q-1)$ -jet of the Poincaré–Takens reduction of (32). The result is that the coefficients  $\ell_m$  for  $m = 0, 1$  are given by

$$\ell_0(\kappa) = \kappa_1 + \frac{i}{2\sqrt{(\kappa_2)_0}}\tilde{\kappa}_2 + O(\varepsilon), \quad (38)$$

$$\begin{aligned} \operatorname{Re}(\ell_1(\kappa)) = & -\frac{(\sqrt{(\kappa_2)_0} - 1)^2 \sqrt{(\kappa_2)_0}((\kappa_2)_0 + 1)}{6(\sqrt{(\kappa_2)_0} + 1)^4} \\ & + \frac{3(\kappa_2)_0}{2((\kappa_2)_0 - 1)}\kappa_4 + O(\kappa_1, \tilde{\kappa}_2, \varepsilon), \end{aligned} \quad (39)$$

$$\operatorname{Im}(\ell_1(\kappa)) = \frac{3(\kappa_2)_0^{3/2}}{2 - 2(\kappa_2)_0}\kappa_3 + O(\kappa_1, \tilde{\kappa}_2, \varepsilon), \quad (40)$$

where  $\tilde{\kappa}_2 = \kappa_2 - (\kappa_2)_0$  with  $(\kappa_2)_0 = \frac{4}{49}\pi^2$ . As regards the recognition conditions, we observe that  $\ell_0(\kappa) = O(\varepsilon)$  and  $\ell_1(\kappa) = O(\varepsilon)$  if

$$\kappa_1 = \tilde{\kappa}_2 = \kappa_3 = 0 \quad \text{and}$$

$$\kappa_4 = (\kappa_4)_0 = \frac{(\sqrt{(\kappa_2)_0} - 1)^3((\kappa_2)_0 + 1)}{9(\sqrt{(\kappa_2)_0} + 1)^3\sqrt{(\kappa_2)_0}}$$

and that  $\operatorname{Re}(\ell_1(\kappa))$  is linear in  $\tilde{\kappa}_4 = \kappa_4 - (\kappa_4)_0$ . Next, we provide  $\ell_2$  near parameter values for which  $\ell_m(\kappa) = O(\varepsilon)$ , with  $m = 0, 1$ :

$$\begin{aligned} \ell_2(\kappa) = & \frac{\sqrt{(\kappa_2)_0}}{108(\sqrt{(\kappa_2)_0} + 1)^{12}} \left( 5(\kappa_2)_0^6 + (32 + 4i)(\kappa_2)_0^{11/2} \right. \\ & + 14(\kappa_2)_0^5 + (160 - 36i)(\kappa_2)_0^{9/2} + (251 - 64i)(\kappa_2)_0^4 \\ & + (448 - 40i)(\kappa_2)_0^{7/2} + 484(\kappa_2)_0^3 + (448 + 40i)(\kappa_2)_0^{5/2} \\ & + (251 + 64i)(\kappa_2)_0^2 + (160 + 36i)(\kappa_2)_0^{3/2} + 14(\kappa_2)_0 \\ & \left. + (32 - 4i)\sqrt{(\kappa_2)_0} + 5 \right) + O(\kappa_1, \tilde{\kappa}_2, \kappa_3, \tilde{\kappa}_4, \varepsilon). \end{aligned} \quad (41)$$

The coefficient  $\gamma_\kappa$  is given by

$$\gamma_\kappa \approx \varepsilon(-83.2 + 141.2i) + O(\kappa_1, \tilde{\kappa}_2, \kappa_3, \tilde{\kappa}_4, \varepsilon^2). \quad (42)$$

Here we explicitly use that  $(\kappa_2)_0 = \frac{4}{49}\pi^2$ , since the expression for  $\gamma_\kappa$  in terms of  $(\kappa_2)_0$  becomes too long to display. For small  $\varepsilon$  the latter coefficients of the Poincaré–Takens reduction of (32) and Theorem 1 result in Proposition 3.  $\square$

#### 4. Conclusion and future work

The main aim of the current paper is presenting a novel practical procedure for detecting non-degenerate or mildly degenerate Hopf–Neimark–Sacker bifurcations in families of periodically forced oscillators. In particular, the mildly degenerate situation forms a ‘next case’ in the general program for recognizing bifurcations; cf. [3]. We also explain how recognition conditions for HNS families determine the local geometry of the resonance set attached to the central resonant HNS bifurcation point. Moreover, it follows from [6] that a skeleton for the local dynamics is also determined by these conditions. As an illustration, the results are applied in case studies.

An interesting extension of the current paper is formulating the recognition conditions for families of vector fields with a periodic orbit in  $\mathbb{R}^k$ , where  $k \geq 3$ , without assuming that the reduction to the center manifold has been performed already; cf. [3]. Furthermore, higher degeneracies could be studied.

#### Appendix A. Normal form coefficients

This appendix presents the coefficients of the  $(q-1)$ -jet of the Poincaré–Takens reduction of (5) as polynomial expressions in the coefficients of the  $(q-1)$ -jet of (5). We note that the Poincaré–Takens reduction procedure is based on well-known methods involving Lie series and a van der Pol transformation as introduced in [9]; see [13, Appendix B]. We performed the computation using Mathematica [33].

Recall that given the latter jet, i.e.,

$$\begin{cases} \dot{y} = (i\omega + h_{10}(\mu))y + \sum_{k+l=2}^{q-1} h_{kl}(\mu)y^k\bar{y}^l \\ \quad + \varepsilon \sum_{k+l=1}^{q-1} H_{kl}(t; \mu)y^k\bar{y}^l, \\ \dot{t} = 1, \end{cases} \quad (A.1)$$

with  $\omega = 2\pi p/q$  and  $h_{10}(0) = 0$ , the Poincaré–Takens reduced form (6) is given by

$$\dot{z} = zG_\mu(|z|^2) + \gamma_\mu \bar{z}^{q-1} + O(|z|^q).$$

Here  $G_\mu$  is given by

$$G_\mu(|z|^2) = \sum_{m=0}^{\lfloor \frac{q-3}{2} \rfloor} \ell_m(\mu)|z|^{2m}.$$

Following [13], we present the explicit expressions for  $\ell_m$  and  $\gamma_\mu$  up to orders  $O(\varepsilon)$  and  $O(\varepsilon^2)$ , respectively. Below we explain that this suffices for solving the recognition problem for (5), as long as  $\varepsilon$  is small enough.

The coefficients of the first three  $\mathbb{S}^1$ -symmetric terms of  $G_\mu$ . We begin with the first three coefficients of  $G_\mu$  given by

$$\ell_0(\mu) = h_{10}(\mu) + O(\varepsilon) \quad (A.2)$$

$$\begin{aligned} \ell_1(\mu) = & h_{21}(\mu) \\ & + \frac{i}{\omega} \left( h_{20}(\mu)h_{11}(\mu) - |h_{11}(\mu)|^2 - \frac{2}{3}|h_{02}(\mu)|^2 \right) \\ & + O(\varepsilon, |h_{10}(\mu)|) \end{aligned} \quad (A.3)$$

$$\ell_2(0) = h_{32}(\mu) + \frac{i}{\omega}L_1 + \frac{1}{\omega^2}L_2 + \frac{i}{\omega^3}L_3 + O(\varepsilon, |h_{10}(\mu)|), \quad (A.4)$$



where

$$\begin{aligned}
 L_1 &= h_{12}h_{30} + 2h_{11}h_{31} + \frac{4}{3}h_{02}h_{40} - h_{13}\bar{h}_{02} \\
 &\quad - \frac{3}{4}h_{03}\bar{h}_{03} - 2h_{22}\bar{h}_{11} - h_{12}\bar{h}_{12} - \frac{2}{3}h_{02}\bar{h}_{13} \\
 &\quad + h_{31}\bar{h}_{20} - h_{11}\bar{h}_{22}, \\
 L_2 &= -2h_{30}h_{11}^2 + 2\bar{h}_{12}h_{11}^2 + 3h_{12}\bar{h}_{02}h_{11} + \frac{31}{12}h_{02}\bar{h}_{03}h_{11} \\
 &\quad + \frac{5}{2}h_{21}|h_{11}|^2 - h_{30}\bar{h}_{20}h_{11} - \frac{1}{2}h_{20}\bar{h}_{21}h_{11} - \frac{3}{2}\bar{h}_{21}|h_{11}|^2 \\
 &\quad - \bar{h}_{02}\bar{h}_{30}h_{11} - 2h_{12}\bar{h}_{11}^2 - \frac{2}{3}h_{02}h_{20}h_{30} + \frac{1}{2}h_{03}h_{20}\bar{h}_{02} \\
 &\quad + \frac{7}{9}|h_{02}|^2h_{21} + h_{12}h_{20}\bar{h}_{11} + \frac{8}{3}h_{02}h_{30}\bar{h}_{11} - \frac{9}{4}h_{03}\bar{h}_{02}\bar{h}_{11} \\
 &\quad + \frac{2}{3}h_{02}h_{20}\bar{h}_{12} - 2h_{02}\bar{h}_{11}\bar{h}_{12} + \frac{1}{6}h_{02}\bar{h}_{03}\bar{h}_{20} + \frac{3}{2}h_{21}\bar{h}_{11}\bar{h}_{20} \\
 &\quad - \frac{5}{9}|h_{02}|^2\bar{h}_{21}, \\
 L_3 &= 2\bar{h}_{02}h_{11}^3 - 2|h_{11}|^4 + h_{20}|h_{11}|^2h_{11} - \bar{h}_{02}\bar{h}_{20}h_{11}^2 \\
 &\quad + \frac{35}{18}|h_{02}|^2h_{20}h_{11} - \frac{23}{4}|h_{02}|^2|h_{11}|^2 + |h_{20}|^2|h_{11}|^2 \\
 &\quad + \frac{4}{3}h_{02}\bar{h}_{11}^3 - \frac{4}{9}|h_{02}|^4 - 2h_{02}h_{20}\bar{h}_{11}^2 + \frac{2}{3}h_{02}h_{20}^2\bar{h}_{11} \\
 &\quad - \frac{1}{3}|h_{02}|^2|h_{20}|^2 + \frac{7}{18}|h_{02}|^2\bar{h}_{11}\bar{h}_{20}.
 \end{aligned}$$

Here  $h_{jk} = h_{jk}(0)$ . The coefficients  $\ell_m$  for  $m > 2$  are not computed, since those do not appear in the recognition conditions in Theorem 1.

The coefficient of the  $z^{q-1}$  term in the Poincaré–Takens reduction. From [13] it follows that  $\gamma_0$  is given by the coefficient of  $e^{2\pi i p t}$  in the Fourier expansion of  $\tilde{H}_{0(q-1)}(t; 0)$ , i.e.,

$$\gamma_0 = \varepsilon \int_0^1 e^{-2\pi i p t} \tilde{H}_{0(q-1)}(t; 0) dt, \quad (\text{A.5})$$

where the  $q$ -dependent  $\tilde{H}_{0(q-1)}(t; 0)$  are presented below for  $3 \leq q \leq 5$ . (The function  $\tilde{H}_{0(q-1)}(t; 0)$  is the coefficient of  $z^{q-1}$  after the non- $\mathbb{Z}_q$ -symmetric terms have been removed from (5) by the Lie series method given in [13, Appendix B].) For  $q > 5$  we refer the reader to [31]. The expressions for  $\gamma_0$  are shortened by adopting the notation  $\tilde{H}_{kl} = \tilde{H}_{kl}(t; 0)$ ,  $H_{kl} = H_{kl}(t; 0)$  and  $h_{kl} = h_{kl}(0)$ .

- The coefficient  $\tilde{H}_{02}$  is given by

$$\begin{aligned}
 \tilde{H}_{02} &= H_{02} - \frac{i}{3\omega}(3h_{11}H_{01} - h_{02}H_{10} \\
 &\quad - 3H_{01}\bar{h}_{20} + 2h_{02}\bar{H}_{10}) + O(\varepsilon, |h_{10}(\mu)|).
 \end{aligned} \quad (\text{A.6})$$

- The coefficient  $\tilde{H}_{03}$  is given by

$$\tilde{H}_{03} = H_{03} + \frac{i}{\omega}A_1 + \frac{1}{\omega^2}A_2 + O(\varepsilon, |h_{10}(\mu)|), \quad (\text{A.7})$$

where

$$\begin{aligned}
 A_1 &= -\frac{1}{2}h_{12}H_{01} + \frac{1}{2}\bar{h}_{30}H_{01} - h_{11}H_{02} + \frac{1}{4}h_{03}H_{10} \\
 &\quad + \frac{1}{3}h_{02}H_{11} + 2H_{02}\bar{h}_{20} - \frac{3}{4}h_{03}\bar{H}_{10} - \frac{2}{3}h_{02}\bar{H}_{20}, \\
 A_2 &= \frac{2}{9}\bar{H}_{01}h_{02}^2 + \frac{1}{3}h_{20}H_{01}h_{02} - \frac{1}{2}H_{01}\bar{h}_{11}h_{02} \\
 &\quad - \frac{1}{2}H_{10}\bar{h}_{20}h_{02} + \frac{1}{12}h_{11}(3H_{10} - 5\bar{H}_{10})h_{02} \\
 &\quad + \frac{5}{6}\bar{h}_{20}\bar{H}_{10}h_{02} - H_{01}\bar{h}_{20}^2 \\
 &\quad - \frac{1}{2}h_{11}^2H_{01} + \frac{3}{2}h_{11}H_{01}\bar{h}_{20}.
 \end{aligned}$$

- The coefficient  $\tilde{H}_{04}$  is given by

$$\begin{aligned}
 \tilde{H}_{04} &= H_{04} + \frac{i}{\omega}B_1 + \frac{1}{\omega^2}B_2 \\
 &\quad + \frac{i}{\omega^3}B_3 + O(\varepsilon, |h_{10}(\mu)|),
 \end{aligned} \quad (\text{A.8})$$

where

$$\begin{aligned}
 B_1 &= -\frac{1}{3}h_{13}H_{01} + \frac{1}{3}\bar{h}_{40}H_{01} \\
 &\quad - \frac{1}{2}h_{12}H_{02} - h_{11}H_{03} + \frac{1}{5}h_{04}H_{10} + \frac{1}{4}h_{03}H_{11} \\
 &\quad + \frac{1}{3}h_{02}H_{12} + 3H_{03}\bar{h}_{20} + H_{02}\bar{h}_{30} - \frac{4}{5}h_{04}\bar{H}_{10} \\
 &\quad - \frac{3}{4}h_{03}\bar{H}_{20} - \frac{2}{3}h_{02}\bar{H}_{30}, \\
 B_2 &= -\frac{1}{9}H_{20}h_{02}^2 + \frac{2}{9}\bar{H}_{11}h_{02}^2 + \frac{2}{9}h_{21}H_{01}h_{02} \\
 &\quad + \frac{1}{3}h_{20}H_{02}h_{02} + \frac{1}{10}h_{12}H_{10}h_{02} + \frac{1}{4}h_{11}H_{11}h_{02} \\
 &\quad - \frac{2}{3}H_{02}\bar{h}_{11}h_{02} - \frac{5}{6}H_{11}\bar{h}_{20}h_{02} - \frac{1}{3}H_{01}\bar{h}_{21}h_{02} \\
 &\quad - \frac{1}{5}H_{10}\bar{h}_{30}h_{02} + \frac{5}{12}h_{03}\bar{H}_{01}h_{02} - \frac{1}{15}h_{12}\bar{H}_{10}h_{02} \\
 &\quad + \frac{2}{15}\bar{h}_{30}\bar{H}_{10}h_{02} - \frac{5}{12}h_{11}\bar{H}_{20}h_{02} + \frac{3}{2}\bar{h}_{20}\bar{H}_{20}h_{02} \\
 &\quad - 3H_{02}\bar{h}_{20}^2 - \frac{1}{2}h_{11}h_{12}H_{01} + \frac{1}{6}h_{03}h_{20}H_{01} \\
 &\quad - \frac{1}{2}h_{11}^2H_{02} - \frac{1}{3}h_{03}H_{01}\bar{h}_{11} + \frac{7}{6}h_{12}H_{01}\bar{h}_{20} \\
 &\quad + \frac{5}{2}h_{11}H_{02}\bar{h}_{20} - \frac{3}{5}h_{03}H_{10}\bar{h}_{20} + \frac{2}{3}h_{11}H_{01}\bar{h}_{30} \\
 &\quad - \frac{4}{3}H_{01}\bar{h}_{20}\bar{h}_{30} + \frac{1}{20}h_{03}h_{11}(4H_{10} - 11\bar{H}_{10}), \\
 B_3 &= \frac{1}{6}H_{01}h_{11}^3 - H_{01}\bar{h}_{20}h_{11}^2 \\
 &\quad + \frac{1}{20}h_{02}(3\bar{H}_{10} - 2H_{10})h_{11}^2 - \frac{5}{18}h_{02}h_{20}H_{01}h_{11} \\
 &\quad + \frac{4}{9}h_{02}H_{01}\bar{h}_{11}h_{11} + \frac{1}{2}h_{02}H_{10}\bar{h}_{20}h_{11} \\
 &\quad - \frac{3}{4}h_{02}\bar{h}_{20}\bar{H}_{10}h_{11} - H_{01}\bar{h}_{20}^3 - \frac{3}{5}h_{02}H_{10}\bar{h}_{20}^2 \\
 &\quad - \frac{5}{27}h_{02}^2H_{01}\bar{h}_{02} - \frac{8}{45}h_{02}^2H_{10}\bar{h}_{11} + \frac{2}{3}h_{02}h_{20}H_{01}\bar{h}_{20} \\
 &\quad + \frac{1}{18}h_{02}^2\bar{h}_{20}\bar{H}_{01} + \frac{9}{10}h_{02}^2\bar{h}_{20}\bar{H}_{10} - \frac{2}{15}h_{02}^2h_{20}\bar{H}_{10} \\
 &\quad + \frac{4}{15}h_{02}^2\bar{h}_{11}\bar{H}_{10} + \frac{33}{20}h_{03}\bar{h}_{20}\bar{H}_{10} - \frac{1}{12}h_{02}^2\bar{H}_{01}h_{11} \\
 &\quad + \frac{4}{45}h_{02}^2h_{20}H_{10} - h_{02}H_{01}\bar{h}_{11}\bar{h}_{20} + \frac{11}{6}H_{01}\bar{h}_{20}^2h_{11}.
 \end{aligned}$$

Solving the recognition problem with (A.2)–(A.4) and (A.5). Next we explain that the expressions (A.2)–(A.4) and (A.5) are sufficient for solving the recognition problem for (5) as long as  $\varepsilon$  is small enough. We only discuss the non-degenerate case, since the mildly degenerate case is similar.

In Section 3 an approximation of a non-degenerate resonant HNS stratum for the family of periodic oscillators given in (5) is obtained by computing a non-degenerate resonant Hopf stratum for the unperturbed system ( $\varepsilon = 0$ ). This approach is based on the observation that the former stratum ‘persists’ for small  $\varepsilon$ , while remaining close to the latter stratum.

We start with showing that the Hopf stratum ‘persists’. Indeed, all inequalities that hold for  $\varepsilon = 0$  remain valid for  $\varepsilon$  small enough. Moreover, if we assume for the moment that  $\varepsilon$  is a real parameter,

then by [Theorem 1](#) resonant HNS points are part of the set

$$S = \{(\mu, \varepsilon) \in \mathbb{R}^n \times \mathbb{R} \mid \ell_0(\mu, \varepsilon) = 0\}.$$

By [\(A.2\)](#) we have that  $\ell_0(0, 0) = h_{10}(0) = 0$ . Additionally, non-degenerate Hopf points of the unperturbed system satisfy

$$\text{Rank}(D_\mu \ell_0(0, 0)) = 2; \quad (\text{A.9})$$

see [\[13\]](#). Hence, we can apply the implicit function theorem, i.e., there is a splitting of  $\mu$ -parameter space  $\mathbb{R}^n = \mathbb{R}^{n-2} \oplus \mathbb{R}^2$  and a local map  $\phi : \mathbb{R}^{n-2} \times \mathbb{R} \rightarrow \mathbb{R}^2$ , such that

$$S = \text{Graph}(\phi)$$

near  $0 \in \mathbb{R}^n \times \mathbb{R}$ .

From [\(A.2\)](#) it follows that the Hausdorff distance [\[34\]](#) between the Hopf stratum for  $\varepsilon = 0$  and the HNS stratum for  $0 < \varepsilon \ll 1$  is  $O(\varepsilon)$ .

## Appendix B. Transforming the linear part into Jordan normal form

This appendix shows how to transform a typical periodically forced oscillator on  $\mathbb{R}^2 \times \mathbb{R}/\mathbb{Z}$ , like the Duffing–van der Pol system given in [\(16\)](#) or the Volterra–Lotka system given in [\(32\)](#), into the form of [\(1\)](#). This is stated more precisely in the following lemma.

**Lemma 4** ([\[13\]](#)). *Given the forced oscillator*

$$\begin{cases} \dot{y} = f_\mu(y) + \varepsilon F_\mu(y, t), \\ \dot{t} = 1, \end{cases} \quad (\text{B.1})$$

with  $(y, t) \in \mathbb{R}^2 \times \mathbb{R}/\mathbb{Z}$  and given that the equilibrium  $y = y_\mu$  of  $\dot{y} = f_\mu(y)$  undergoes a Hopf bifurcation at  $\mu = 0$  with critical eigenvalues  $\lambda_\pm = \pm 2\pi i \frac{p}{q}$ , then consecutively applying four appropriate transformations, brings system [\(B.1\)](#) into the form of [\(1\)](#), i.e.,

$$\begin{cases} \dot{z} = \left(2\pi i \frac{p}{q} + h_{10}(\mu)\right) z + h_\mu(z) + \varepsilon H_\mu(z, t), \\ \dot{t} = 1, \end{cases} \quad (\text{B.2})$$

where  $(z, t) \in \mathbb{C} \times \mathbb{R}/\mathbb{Z}$ ,  $h_{10}(0) = 0$ ,  $h_\mu(z) = O(|z|^2)$  and  $H_\mu(z, t)$ , which is of order  $O(|z|)$ , depends 1-periodically on  $t$ .

To be able to explicitly compute  $h_\mu$  and  $H_\mu$  from  $f_\mu$  and  $F_\mu$ , the four transformations mentioned in the latter lemma need to be known in more detail; cf. [\[13\]](#).

1. The first transformation is given by  $y \mapsto Y = y - y_\mu$ , which ensures that  $y_\mu$  is translated to the origin. As a result, the system  $\dot{y} = f_\mu(y)$  is transformed into

$$\dot{Y} = L_\mu(Y) + \tilde{f}_\mu(Y), \quad (\text{B.3})$$

where  $L_\mu(Y)$  is linear in  $Y$  and  $\tilde{f}_\mu(Y) = O(|Y|^2)$ .

2. The second transformation identifies  $\mathbb{R}^2 \times \mathbb{R}/\mathbb{Z}$  with  $\mathbb{C} \times \mathbb{R}/\mathbb{Z}$  using the map

$$(Y_1, Y_2, t) \mapsto (Z, \bar{Z}, t) = (Y_1 + iY_2, Y_1 - iY_2, t).$$

As a result, system [\(B.3\)](#) becomes

$$\dot{Z} = \check{L}_\mu(Z) + \check{f}_\mu(Z). \quad (\text{B.4})$$

3. The third transformation  $Z \mapsto \zeta = s_\mu^{-1}Z$  ensures that the linear part of [\(B.4\)](#) is transformed into Jordan normal form. More precisely, without loss of generality we may assume that  $\check{L}_\mu$  is of the form

$$\check{L}_\mu(Z) = (a_\mu + ib_\mu)Z + (c_\mu + id_\mu)\bar{Z}.$$

Then  $s_\mu^{-1} \circ \check{L}_\mu \circ s_\mu(\zeta) = (a_\mu + i\omega_\mu)\zeta$ , with  $\omega_\mu = \sqrt{b_\mu^2 - c_\mu^2 - d_\mu^2}$ , if  $s_\mu$  is given by

$$s_\mu(z) = \sigma_\mu \zeta + \tau_\mu \bar{\zeta},$$

where  $\sigma_\mu = \frac{1}{2}(b_\mu - d_\mu + \omega_\mu + ic_\mu)$  and  $\tau_\mu = \frac{1}{2}(b_\mu - d_\mu - \omega_\mu + ic_\mu)$ , and the map  $s_\mu^{-1}$  is given by

$$s_\mu^{-1}Z = \frac{\bar{\sigma}_\mu Z - \tau_\mu \bar{Z}}{|\sigma_\mu|^2 - |\tau_\mu|^2}.$$

We note that so far [\(B.1\)](#) has been transformed to the form

$$\begin{cases} \dot{\zeta} = (i\omega_\mu + a_\mu)\zeta + \hat{f}_\mu(\zeta) + \varepsilon \hat{F}_\mu(\zeta, t), \\ \dot{t} = 1. \end{cases}$$

By assuming that  $\lambda_\pm = \pm 2\pi i \frac{p}{q}$  are critical eigenvalues of the  $\dot{\zeta}$ -component of the system, it follows that  $i\omega_0 + a_0 = \lambda_+$ . So  $h_{10}(\mu)$  in [\(B.2\)](#) is given by

$$h_{10}(\mu) = i\omega_\mu + a_\mu - \lambda_+.$$

4. The last transformation  $\zeta \mapsto z = \zeta + \varepsilon \phi_\mu(t)$  ensures that the periodic forcing term is of order  $O(|z|)$ . Indeed, using that

$$\begin{aligned} \dot{z} &= \dot{\zeta} + \varepsilon \dot{\phi}_\mu(t) \\ &= (i\omega_\mu + a_\mu)\zeta + \hat{f}_\mu(\zeta) + \varepsilon(\hat{F}_\mu(\zeta, t) + \dot{\phi}_\mu(t)) \\ &= \varepsilon(\hat{F}_\mu(0, t) - (i\omega_\mu + a_\mu)\phi_\mu(t) + \dot{\phi}_\mu(t)) + O(|z|), \end{aligned}$$

we see that  $\dot{z}$ , and thus the periodic forcing  $H_\mu(z, t)$  in [\(B.2\)](#), is of order  $O(|z|)$  if  $\phi_\mu(t)$  is chosen such that it solves

$$\dot{\phi}_\mu(t) - (i\omega_\mu + a_\mu)\phi_\mu(t) + \hat{F}_\mu(0, t) = 0.$$

**Example:** The Duffing–van der Pol oscillator for  $(p, q) = (1, 5)$ . As an example, we apply the latter transformations to the Duffing–van der Pol oscillator

$$\begin{cases} \dot{u} = v, \\ \dot{v} = -(v_1 + v_3 u^2)v - v_2 u - v_4 u^3 - u^5 + \varepsilon(1 + u^6) \cos(2\pi t), \\ \dot{t} = 1. \end{cases}$$

It is assumed the system undergoes a non-degenerate 1:5-resonant HNS bifurcation. Note that the multi-parameter on which this system depends is called  $v$  here instead of  $\mu$  as it was in [Lemma 4](#).

By applying the transformations following [Lemma 4](#) to the Duffing–van der Pol system, a system of the form [\(B.2\)](#) is obtained for which the coefficient  $h_{10}(v)$  is given by

$$h_{10}(v) = \frac{1}{2} \left( -v_1 - \sqrt{v_1^2 - 4v_2} \right).$$

Moreover, if we write the series expansion of  $h_v(z)$  in [\(B.2\)](#) as  $\sum_{2 \leq j+k} h_{jk}(v) z^j \bar{z}^k$ , then the non-zero  $h_{jk}$  with  $2 \leq j+k \leq 3$  are given by

$$h_{30}(0, v_2, v_3, v_4) = -\frac{v_3 + \frac{iv_4}{\sqrt{v_2}}}{2(\sqrt{v_2} - 1)^2},$$

$$h_{21}(0, v_2, v_3, v_4) = -\frac{v_3 + \frac{3iv_4}{\sqrt{v_2}}}{2 - 2v_2},$$

$$h_{12}(0, v_2, v_3, v_4) = \frac{v_3 - \frac{3iv_4}{\sqrt{v_2}}}{2(\sqrt{v_2} + 1)^2},$$

$$h_{03}(0, v_2, v_3, v_4) = -\frac{(\sqrt{v_2} - 1)^4 (\sqrt{v_2} v_3 - iv_4)}{2(v_2 - 1)^3 \sqrt{v_2}}.$$

To solve the recognition problem for non-degenerate 1:5-resonance only these coefficients are needed for the given parameter values; see [Remarks 1](#) and [Section 3](#).

Finally, if we write the series expansion of  $H_v(z, t)$  as  $\sum_{j+k} H_{jk}(v, t) z^j \bar{z}^k$ , then up to order  $O(\varepsilon)$  the non-zero coefficients  $H_{jk}(v)$  with  $1 \leq j+k \leq 2$  are given by the equations in [Box 1](#).

$$\begin{aligned}
H_{20} &= \frac{1}{4(\sqrt{\nu_2} - 1)^2(\nu_2 - 4\pi^2)} \left( \frac{3(\sqrt{\nu_2} - 1)(\nu_4 - i\sqrt{\nu_2}\nu_3)(i\sqrt{\nu_2}\cos(2\pi t) + 2\pi\sin(2\pi t))}{\sqrt{\nu_2^2}} \right. \\
&\quad \left. - \frac{(\sqrt{\nu_2} - 1)^2(\sqrt{\nu_2}\nu_3 + 3i\nu_4)(\sqrt{\nu_2}\cos(2\pi t) + 2i\pi\sin(2\pi t))}{\nu_2^{3/2} + \nu_2} \right), \\
H_{11} &= \frac{1}{2(\nu_2 - 4\pi^2)} \left( \frac{i(\sqrt{\nu_2} - 1)(\sqrt{\nu_2}\nu_3 + 3i\nu_4)(i\sqrt{\nu_2}\cos(2\pi t) + 2\pi\sin(2\pi t))}{(\nu_2 - 1)\sqrt{\nu_2^2}} \right. \\
&\quad \left. - \frac{(\sqrt{\nu_2} - 1)(\sqrt{\nu_2}\nu_3 - 3i\nu_4)(\sqrt{\nu_2}\cos(2\pi t) + 2i\pi\sin(2\pi t))}{(\sqrt{\nu_2} + 1)^2\nu_2} \right), \\
H_{02} &= \frac{1}{4(\nu_2 - 4\pi^2)} \left( \frac{3(\sqrt{\nu_2}\nu_3 - i\nu_4)(\sqrt{\nu_2} - 1)^5(\sqrt{\nu_2}\cos(2\pi t) + 2i\pi\sin(2\pi t))}{(\nu_2 - 1)^3\nu_2} \right. \\
&\quad \left. + \frac{(\sqrt{\nu_2} - 1)(3\nu_4 + i\sqrt{\nu_2}\nu_3)(i\sqrt{\nu_2}\cos(2\pi t) + 2\pi\sin(2\pi t))}{(\sqrt{\nu_2} + 1)^2\sqrt{\nu_2^2}} \right).
\end{aligned}$$

### Box I.

Here the coefficients  $H_{jk} = H_{jk}(0, \nu_2, \nu_3, \nu_4, t)$  are given up to order  $O(\varepsilon)$ . Again only these are of interest for the recognition problem.

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